

## Chapter 2 Solutions of Equation of One Variable

Find all roots of the equation  $f(x) = 0$

### § Bisection method

1. Try two distinct values  $a$  and  $b$  to satisfy  $f(a)f(b) < 0$ . Then one root  $x^*$  locates inside  $(a, b)$ .

2. Let  $p_1 = a$ ,  $p_2 = b$

3.  $p_{i+2} = (p_i + p_{i+1}) / 2$ .

If  $|f(p_{i+2})| < \varepsilon$  then  $p_{i+2}$  is one of the roots and stop

If  $f(p_i)f(p_{i+3}) < 0$ , then  $p_{i+1} = p_{i+2}$ , otherwise  $p_i = p_{i+2}$

Go to step3.

Property:  $|p_i - x^*| < \frac{b-a}{2^i}$ .

### § Method of false position

1. Try two distinct values  $p_0$  and  $p_1$  to satisfy  $f(p_0)f(p_1) < 0$ . Then one root  $p^*$  locates inside  $(p_0, p_1)$ .

2. Connect two points  $(p_0, f(p_0))$  and  $(p_1, f(p_1))$  to form a line.

Find the intersection  $(p_2, 0)$  of the line and the x-axis:

$$\frac{0 - f(p_0)}{p_2 - p_0} = \frac{f(p_1) - f(p_0)}{p_1 - p_0} \quad \text{or} \quad p_2 = p_0 - \frac{(p_1 - p_0)f(p_0)}{f(p_1) - f(p_0)}$$

If  $|f(p_2)| < \varepsilon$ , then  $p^* = p_2$  and stop, otherwise

3. If  $f(p_2)f(p_1) < 0$ , then  $p_0 = p_2$  otherwise  $p_1 = p_2$ .

Go to step 2.

## § Newton's method

Expand  $f(x)$  about  $x^*$  into the following form

$$f(x) = f(x^*) + f'(x^*)(x - x^*) = f'(x^*)(x - x^*)$$

and solve  $x^*$  as

$$x^* = x - f(x) / f'(x^*)$$

which can be approximated as  $x_{i+1} = x_i - f(x_i) / f'(x_i)$

## § Secant method

$$\text{Express } f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})f(x_i)}{f(x_i) - f(x_{i-1})}$$

## § Rate of convergence

Suppose  $\{p_i\}_{i=1} \rightarrow p$

1. Linearly converge  $|p - p_{i+1}| \leq M |p - p_i|$

2. Quadratically converge  $|p - p_{i+1}| \leq M |p - p_i|^2$

The Newton's method has a quadratically convergent rate.

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})f(x_i)}{f(x_i) - f(x_{i-1})}.$$

## § Aitken's method

$$p_{n+1} = g(p_n)$$

Suppose  $\{p_i\}_{i=1} \rightarrow p$  to be linearly convergent, then

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}$$

which can be solved for  $p$  as

$$p = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Introduce  $\{q_i\}_{i=1}$

$$q_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \dots \text{Aitken's method}$$

which has the property

$$\lim_{n \rightarrow \infty} \frac{q_n - p}{p_n - p} = 0$$

§ Steffensen's method

Modify Aitken's method

1. Given an initial value  $p_0^{(0)}$

1. Performing  $p_1^{(i)} = g(p_0^{(i)})$  and  $p_2^{(i)} = g(p_1^{(i)})$

$$p_0^{(i+1)} = p_0^{(i)} - \frac{(p_1^{(i)} - p_0^{(i)})^2}{p_2^{(i)} - 2p_1^{(i)} + p_0^{(i)}}$$

If  $|p_0^{(i+1)} - p_0^{(i)}| < \varepsilon$  output  $p_0^{(i)}$  and stop

Otherwise stop for  $i > N$ .

§ Müller's method

The complex roots of the equation  $f(x) = 0$  cannot be solved by the

previously mentioned methods. If  $f(x) = 0$  has a pair of complex roots

$r + is$  and  $r - is$ , then

$f(x) = 0$  can be decomposed as

$$f(x) = 0 = (x - r - is)(x - r + is)q(x) = [x^2 - 2rx + (r^2 + s^2)]q(x).$$

Therefore, three points are needed to generate a polynomial of order two

to approximate  $P_2(x) = x^2 - 2rx + (r^2 + s^2)$ .

Try  $\{p_0, f(p_0)\}$ ,  $\{p_1, f(p_1)\}$  and  $\{p_2, f(p_2)\}$  to generate the quadratic form

$$\hat{P}_2(x) = \frac{(p-p_1)(p-p_2)}{(p_0-p_1)(p_0-p_2)} f(p_0) + \frac{(p-p_0)(p-p_2)}{(p_1-p_0)(p_1-p_2)} f(p_1) + \frac{(p-p_0)(p-p_1)}{(p_2-p_0)(p_2-p_1)} f(p_2)$$

and find the root  $p_3$  of the equation  $\hat{P}_2(x) = 0$ . Then employing

$\{p_1, f(p_1)\}$  and  $\{p_2, f(p_2)\}$ , and  $\{p_3, f(p_3)\}$  into similar procedures of calculation until the convergence is obtained.

Ex.  $f(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6$

$$p_0 = 0.5, \quad p_1 = -0.5, \quad p_2 = 0$$

---

| n | $p_n$                   | $f(p_n)$                                              |
|---|-------------------------|-------------------------------------------------------|
| 3 | $-0.555556 + 0.598352i$ | $-29.4007 - 3.89872i$                                 |
| 4 | $-0.435450 + 0.102101i$ | $1.33223 - 1.19309i$                                  |
| 5 | $-0.390631 + 0.141852i$ | $0.375057 - 0.670164i$                                |
| 6 | $-0.357699 + 0.169926i$ | $-0.146746 - 0.00744629i$                             |
| 7 | $-0.356051 + 0.162856i$ | $-0.183868 \times 10^{-2} + 0.539780 \times 10^{-3}i$ |
| 8 | $-0.356062 + 0.162758i$ | $0.286102 \times 10^{-5} + 0.953674 \times 10^{-6}i$  |

$$p_0 = 0.5, \quad p_1 = 1.0, \quad p_2 = 1.5$$

---

| n | $p_n$   | $f(p_n)$                  |
|---|---------|---------------------------|
| 3 | 1.28785 | -1.37624                  |
| 4 | 1.23746 | 0.126941                  |
| 5 | 1.24160 | $0.219440 \times 10^{-2}$ |
| 6 | 1.24168 | $0.257492 \times 10^{-4}$ |
| 7 | 1.24168 | $0.257492 \times 10^{-4}$ |

$$p_0 = 2.5, \quad p_1 = 2.0, \quad p_2 = 2.25$$

---

| n | $p_n$   | $f(p_n)$                   |
|---|---------|----------------------------|
| 3 | 1.96059 | -0.611255                  |
| 4 | 1.97056 | $0.748825 \times 10^{-2}$  |
| 5 | 1.97044 | $-0.295639 \times 10^{-4}$ |
| 6 | 1.97044 | $-0.295639 \times 10^{-4}$ |

## HW2

1. The following four methods are proposed to compute  $9^{1/5}$ . Rank them in order based on their apparent speeds of convergence, assuming

$$p_0 = 1.$$

a.  $p_n = \left(1 + \frac{9 - p_{n-1}^3}{p_{n-1}^2}\right)^2$ , b.  $p_n = p_{n-1} - \frac{p_{n-1}^5 - 9}{p_{n-1}^2}$  c.  $p_n = p_{n-1} - \frac{p_{n-1}^5 - 9}{5p_{n-1}^4}$

d.  $p_n = p_{n-1} - \frac{p_{n-1}^5 - 9}{12}.$

3. Determine all the real roots of  $f(x) = -26 + 85x - 91x^2 + 44x^3 - 8x^4 + x^5$  by

(a) Bisection method, (b) False-position method, (c) Newton's method, (d) Secant method, and (e) Steffensen's method.

4. The equation  $f(x) = x^4 - 5x^3 + 18x^2 - 34x + 20 = 0$  has two real roots and a pair of complex roots. Find all roots of the equation  $f(x) = 0$ .

5. Find approximations to within  $10^{-5}$  to all zeros of the following polynomial by first finding the real zeros using Newton's method and then reducing to a polynomial of lower degree to determine any complex zeros.  $f(x) = x^5 + 11x^4 - 21x^3 - 10x^2 - 21x - 5$